Equilibrium-Based Pricing in Financial Markets

by

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A thesis submitted in conformity with the requirements

for the degree of Master of Science

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March, 2018

Disclaimer

I, Cristian Viorel Hrenciuc, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

This project focuses on the way equilibrium prices are determined in discrete time. We will consider a two-period model and a multi-period model, under the assumption of different market structures, where aspects related to completeness and Pareto optimality will be analyzed. An important result of this project was the separation of the mathematics from the financial economics.

This thesis was completed under the supervision of ***Dr. Daniel C. Schwarz****.*

Acknowledgments

Firstly I would like to thank my advisor, Dr. Daniel C. Schwarz, for his guidance and infinite patience. The project would still be in its infancy without your advice.

Many thanks to my parents, Alina Dan and Stelu Viorel Hrenciuc, and my girlfriend, Maria Georgiana Simionescu, for their continuous support throughout my years of study.

Last, but not least, I would like to thank Aditya Vikram Singh, who brought a bit of boisterousness in times of intense research.

Cristian Viorel Hrenciuc, *University College London*, March 2018

Mathematics is the only good metaphysics.

William Thomson, 1st Baron Kelvin

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Chapter 1

Introduction

The field of economics has vastly changed through the works of Keynes and the combined efforts of Arrow and Debreu. While the former introduced assumptions related to the inherent imperfections of markets, thus creating what is considered modern economics [1], Arrow and Debreu formalized the concept of contingent market under uncertainty and proved the existence of a contingent market equilibrium [2]. Magill and Quinzii expanded on this basis and extended the model to a setting of several periods, in order to better quantify the gradual unfolding of information [3], and it is their work in incomplete markets that this project is based on. We shall be focusing our attention on the case of a single good economy.

Throughout the following, we shall refer to a contingent contract (or a contingent claim) as to a contract that has a payoff of one good if a specific event happens. The utility function and the wealth endowment of an agent are considered exogenous functions. A contingent market equilibrium consists of a set of prices for the contingent claims and consumption streams that enable each investor to maximize their utility. The financial market equilibrium is a similar concept, but we allow for more complex securities. All such notions will be formally defined and explained, for a two-period setting in Chapter 3 and for a multi-period setting in Chapter 4.

Some of the results presented here can be found in the literature, but it is in this project that one can observe a detailed presentation of the 'mathematics', in terms of the relevant fixed-point theorem, separated from the 'economics', where equilibria are presented and their existence is proved. Chapter 2 is a self-contained, detailed explanation of the fixed-point theorem that is relevant for the financial analysis and it contains no reference to any economics theory, thus making it accessible to researchers with little or no experience with financial economics.

Notation

Throughout this paper, $f(A)$ will denote the image of the function f under the set A. The boundary of a set S is defined as the closure of S without the interior of S and we shall write it as ∂S.

For $x, y \in \mathbb{R}^n$,

$$x \geq y \Longleftrightarrow x_j \geq y_j, \forall j = 1, ..., n,$$

$$x > y \Longleftrightarrow x_j > y_j, \forall j = 1, ..., n.$$

We will be using $\|x\|$ to define the Euclidean norm, for $x = (x_1, x_2, ..., x_n)$, as such:

$$\|x\| := \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

$\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ will denote, in turn, the space of all non-negative and positive n-dimensional vectors, that is:

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n | x_i \geq 0 \ \forall i\}, \quad \mathbb{R}^n_{++} := \{x \in \mathbb{R}^n | x_i > 0 \ \forall i\}.$$

For the purposes of our discussion, we define some elementary concepts: let B^n denote the n-dimensional unit ball and S^{n-1} the $(n-1)$-dimensional unit sphere, that is,

$$B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}, \quad S^n := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Further, let S_+^{n-1} and S_{++}^{n-1} denote the non-negative and positive unit spheres:

$$S_+^{n-1} := S^{n-1} \cap \mathbb{R}_+^n, \quad S_{++}^{n-1} := S^{n-1} \cap \mathbb{R}_{++}^n.$$

Chapter 2

Fixed-Point Theorems in Financial Economics

From a purely mathematical point of view, constructing an equilibrium in a financial market can be reduced to a fixed-point problem. For this reason we begin with a discussion of fixed-point theory and prove a version of a fixed-point theorem that is suitable for our later applications to equilibrium theory.

Conceptually, a fixed point of a function is an element in the domain of the function which is invariant under the mapping provided by the function. In other words, a point is a fixed point of a function if it is mapped to itself. For the convenience of the reader we provide the following definition:

Definition 1 (Fixed Point) *Let $v = v(y) : A \subseteq \mathbb{R}^n \mapsto A$. Then y_0 is a fixed point of v if $v(y_0) = y_0$.*

A fixed-point theorem of fundamental importance in financial economics is Brouwer's fixed-point theorem for real-valued functions v with domain and codomain $A \subset \mathbb{R}^n$.

Theorem 1 (Brouwer's Fixed-Point Theorem) *Let $v = v(y) : A \subset \mathbb{R}^n \mapsto A$. If the map $y \mapsto v(y)$ is continuous and A is non-empty, convex and compact then v has a fixed point.*

Proof The proof is not given here, but can be found by the interested reader in any textbook on fixed-point theory or functional analysis, for example Section 17.2 in [4]. ∎

Remark 1 *Brouwer's fixed-point theorem only guarantees existence, not uniqueness.*

Brouwer's theorem is not directly applicable to our task of constructing equilibria in financial economics. This is due to the fact that, when in a financial environment, both the contingent market equilibrium and the financial market equilibrium are defined as vector fields on the positive unit sphere, which is not a convex set and thus violates the assumptions for Brouwer's fixed-point theorem.

For this reason we establish a different fixed-point theorem, one which will be directly applicable to the existence proofs of equilibria later on. The fixed-point theorem for equilibrium analysis will differ from Brouwer's by the fact that several extra assumptions are imposed upon the behaviour of the function, which will enable us to define a fixed-point theorem under these rather specific conditions, but without the requirement of a convex set.

We begin with the definition of an inward-pointing vector field:

Definition 2 (Inward-Pointing Vector Field) *A vector field* $v = v(y) : S_+^{n-1} \mapsto \mathbb{R}^n$ *is inward-pointing on the boundary* ∂S_+^{n-1} *if* $y \cdot v(y) \leq 0, \ \forall y \in \partial S_+^{n-1}$.

Let $v : S_{++}^{n-1} \mapsto \mathbb{R}^n$ be an inward-pointing vector field, satisfying the following assumptions:

$(A1)$ (boundedness from below) there exists $a \in \mathbb{R}^n$ such that $v(y) \geq -a$,

$(A2)$ (continuity) the map $y \mapsto v(y)$ is continuous,

$(A3)$ (positive homogeneity) $v(ay) = v(y)$ for any constant $a \in \mathbb{R}_{++}$,

$(A4)$ (Walras' Law) $y \cdot v(y) = 0$ for all $y \in S_{++}^{n-1}$,

$(A5)$ (boundary behaviour) If $y_n \in S^{n-1}_{++}$ such that $y_n \to y \in \partial S^{n-1}_{++}$, $y \neq \mathbf{0}$, then $\|v(y_n)\| \to \infty$ as $n \to \infty$.

Lemma 1 *Under $(A1)-(A5)$ we have an alternative definition of an inward-pointing vector field: A vector field $v = v(y) : S^{n-1}_{+} \mapsto \mathbb{R}^n$ is inward-pointing on the boundary ∂S^{n-1}_{+} if $v_j(y) \geq 0$ for all $y \in \partial S^{n-1}_{+}$ with $y_j = 0$.*

Proof Consider $v(y) : S^{n-1}_{+} \mapsto \mathbb{R}^n$, $y \cdot v(y) = \sum_{j=1}^{n} y_j \cdot v_j(y)$. Define $J := \{v_j(y) \mid v_j(y) \geq 0,\ j = 1, ..., n\}$. For all $v_j(y) \in J$ we have $y_j = 0$, thus

$$\sum_{v_j(y) \in J} y_j \cdot v_j(y) = 0.$$

For $v_j(y) \notin J$, that is for $v_j(y) < 0$, we have $y_j \geq 0$ by the fact that $y \in S^{n-1}_{+}$, so $\sum_{v_j(y) \notin J} y_j \cdot v_j(y) \leq 0$. Moreover, the point $\mathbf{0} = (0, 0, ..., .) \notin S^{n-1}_{+}$, so we must have at least one positive entry y_j. This results in the fact that there exists at least one $j \in \{1, ..., n\}$ such that $y_j \cdot v_j(y) < 0$, thus

$$\sum_{v_j(y) \notin J} y_j \cdot v_j(y) < 0.$$

To sum it up,

$$y \cdot v(y) = \sum_{v_j(y) \in J} y_j \cdot v_j(y) + \sum_{v_j(y) \notin J} y_j \cdot v_j(y) < 0,$$

therefore the definition presented in Lemma 1 is equivalent to the one presented in Definition 2, under assumptions $(A1) - (A5)$. ∎

Given that it will be referenced later on, we introduce the Inward-Pointing Vector Field Theorem here. This statement is presented as Theorem 7.5 in [3].

Theorem 2 (Inward-Pointing Vector Field Theorem) *If $f = f(y) : S^{n-1}_{+} \mapsto$*

$\mathbb{R}^n$ *is a continuous inward-pointing vector field, then there exists* $y_0 \in S_+^{n-1}$ *such that* $f(y_0) = \boldsymbol{0}$.

Consider a vector field $v = v(y) : S_{++}^{n-1} \mapsto \mathbb{R}^n$. The following is the main result of this section:

Theorem 3 (Fixed-Point Theorem for Equilibrium Analysis) *Let* v *satisfy (A1)-(A5), then there exists* y_0 *such that* $v(y_0) = \boldsymbol{0}$.

The proof of Theorem 3 is lengthy and requires several non-trivial lemmas to be presented beforehand, hence we shall give a general (informal) overview, as well as the motivation for progressing to the next step, while the proof will be written in a constructive manner, clarifying and proving each lemma before using it.

Our goal is to prove that a vector field which satisfies the given assumptions has a zero. If we look at Theorem 2 and Theorem 3 we notice some similarities between the properties of the vector field at hand. Maintaining the same notation, we can see the 'slight' difference of domain (**non-negative** unit sphere for $f(y)$ and **positive** unit sphere for $v(y)$). In a sense, $v(y)$ is 'almost' inward-pointing, with the exception of the sphere's boundary where it increases without limits. Hence, we require $(A1) - (A4)$, which are assumptions that guarantee no zero of v is situated in the proximity of the boundary.

Thus, what we need to do is find an inward-pointing vector field $\widehat{v} : S_+^{n-1} \mapsto \mathbb{R}^n$ defined on the non-negative sphere that has the same zeros as $v : S_{++}^{n-1} \mapsto \mathbb{R}^n$, and then use the Inward-Pointing Vector Field Theorem on $\widehat{v}$. We define $\widehat{v}$ as the weighted sum of v and v^*, with α later defined as the weighting function, where v^* is a suitable inward-pointing vector field, with the purpose of proving that $\widehat{v}$ and v share the same zeros.

We will use the following structure for the proof: the first two lemmas will be dedicated to proving the existence of α, then we present a lemma about the codomain of α and the last lemma is the proof of existence of $\widehat{v}$. We conclude with the actual proof of Theorem 3, which will use all the results introduced earlier.

An important aspect to keep in mind throughout the following is the fact that all functions used to prove the existence of α are continuous (f_{ab}, g, g_k, ε, β and α itself). We will emphasize the end-points (where the function is equal to either 0 or 1), but one must always keep in mind the mid-points, where the value of the functions belongs to $(0, 1)$.

Let us first introduce the function ε, which maps elements from an $(n-1)$-dimensional hyperball to the interval $[0, 1]$.

Lemma 2 *Let E and F be subsets of $\mathbb{R}^{n-1}$ with E compact, F closed and $E \cap F = \oslash$. Then there exists a continuous function $\varepsilon : \mathbb{R}^{n-1} \mapsto [0, 1]$ such that:*

$$\varepsilon(y) = \begin{cases} 1, & \textit{if } y \in E, \\ 0, & \textit{if } y \in F. \end{cases}$$

Proof For a, b real numbers such that $0 \leq a \leq b$, let $f_{ab} : \mathbb{R} \Rightarrow [0, 1]$ be defined as:

$$f_{ab}(t) := \begin{cases} 1, & \text{if } t \in [0, a], \\ 1 - \frac{t-a}{b-a}, & \text{if } t \in (a, b), \\ 0, & \text{if } t \geq b. \end{cases}$$

Let $B(y_0, a)$ and $B(y_0, b)$ define real $(n-1)$-dimensional concentric balls such that they are centered at y_0 and have radius a and b, respectively. We also assume $B(y_0, a)$ is not degenerate, in the sense that it is not a point (i.e $a > 0$).

Let $g : \mathbb{R}^{n-1} \Rightarrow [0, 1]$, $g(y) := f_{ab}(\|y - y_0\|)$. Thus,

$$g(y) = \begin{cases} 1, & \text{if } y \in B(y_0, a), \\ 1 - \frac{\|y-y_0\|-a}{b-a}, & \text{if } y \in B(y_0, b) \setminus B(y_0, a), \\ 0, & \text{if } y \notin B(y_0, b). \end{cases}$$

E is compact, which means, by definition of compactness, that we can set it as a finite union of closed hyperspaces. Thus we can cover it with a finite number of hyperballs B_k, for $k = 1, ..., m$. F is closed, thus the complement of F is open, which means we can choose the hyperballs B_k such that $B_k \cap F = \oslash$, for any k. By the property of an open covering, we can choose $\widehat{B}_k \subset B_k$, for any k, such that we still get $E \subset \cup_{k=1}^m \widehat{B}_k$. Thus, there exist functions g_k such that:

$$g_k(y) = \begin{cases} 1, & \text{if } y \in B_k, \\ 0, & \text{if } y \notin \widehat{B_k}. \end{cases}$$

Define

$$\varepsilon(y) := 1 - \prod_{k=1}^{m} (1 - g_k(y)).$$

There are three options regarding the behaviour of y in relation to E and F.

($y \in E$) If y is in E then we must have that $y \in \widehat{B}_k$ for at least one k. In turn, the functions $g_k(y)$ for which $y \in \widehat{B}_k$ are equal to 1, which makes the product equal to 0. In this case, we have $\varepsilon(y) = 1$.

($y \in F$) If y belongs to F then y does not belong to any hyperball B_k, which means that $g_k(y) = 0$ for all k. In this case, we have $\varepsilon(y) = 0$.

($y \notin E \cup F$) We refer back to $g(y)$, a function that measures the Euclidean norm from y_0, the center of the concentric balls defined at the beginning of the proof. If $y \notin E \cup F$ then $g(y)$ falls into the second category, where its value lies in the interval $(0, 1)$. This property translates further to the functions $g_k(y)$, but in the interval $(0, 1]$. If $y \notin E \cup F$ there exists at least one k for which $g_k(y) \neq 1$, hence $\varepsilon(y) \neq 1$. In this case, we have $\varepsilon(y) \in (0, 1)$.

Note that in this case we allow for $y \in B_k$, but not for $y \in \widehat{B}_k$.

The continuity of ε is assured by the third case and by the fact that it is a product of continuous functions. Thus, the function $\varepsilon(y)$ satisfies the required properties. ∎

We must now find a way to change the image from the non-negative unit sphere to the real numbers, in $n-1$ dimensions. We do this by defining a function, β, such that α is ε composed with β. This will allow us to have a weighting function α that has the properties which will be required later on.

Lemma 3 *Let K and U be subsets of S_+^{n-1} with K closed, U open, where $K \subset U \subset \bar{U}$. Then there exists a continuous function $\alpha = \alpha(y) : S_+^{n-1} \mapsto [0, 1]$, such that:*

$$\alpha(y) = \begin{cases} 1, & \text{if } y \in K, \\ 0, & \text{if } y \notin U. \end{cases}$$

Proof Let $\beta : S_+^{n-1} \mapsto \mathbb{R}^{n-1}$ be such that, for any $y = (y_1, ..., y_n)$, $\beta(y) := (y_1, ..., y_{n-1})$. Thus, β is a projection from n dimensions to $n-1$. Moreover, it maps elements from the $(n-1)$-dimensional non-negative unit sphere to the $(n-1)$-dimensional non-negative unit ball, as

$$\beta(S_+^{n-1}) = \{y \in \mathbb{R}_+^{n-1} \mid \|y\| \leq 1\} = B_+^{n-1}.$$

Furthermore, we define E as $\beta(K)$ and F as $\beta(S_+^{n-1}) \setminus \beta(U)$. Thus, $\alpha(y) := \varepsilon(\beta(y))$. ∎

In order to make the use of α applicable to the proof of Theorem 3, we need to use a specific set K. The following lemma defines such a set, but the fact that it is closed is non-trivial. To have consistency with the definition of α, the fact that the given K is closed will be proven explicitly.

Lemma 4 *Let $v : S_{++}^{n-1} \mapsto \mathbb{R}^n$ satisfy $(A1) - (A5)$ and let $N_j := \{y \in S_{+}^{n-1} | v_j(y) > 0, y_j < 1/\sqrt{n}\}$. We define $K := S_{++}^{n-1} \setminus N$ for $N = \cup_{j=1}^{n} N_j$. Then K is closed as a subset of S_{+}^{n-1}.*

Proof Assume that $y \in \partial S_{+}^{n-1}$, which means that there exists j such that $y_j = 0$. Let $y_m \in K$ be a sequence such that $y^m \to y$. $\partial S_{+}^{n-1} \not\subset S_{++}^{n-1} \Rightarrow y^m \notin N \Rightarrow v_j(y^m) \leq 0$ ($v_j(y)$ is upper bounded). From $(A1)$ we have $v_j(y) \geq -a_j$ ($v_j(y^m)$ is lower bounded). Since $v_j^m(y)$ is bounded for all j then $\|v(y^m)\|$ is also bounded, which contradicts $(A5)$.

Thus, $y \notin \partial S_{+}^{n-1}$, so $y \in S_{++}^{n-1}$. N is trivially open and since $y^m \notin N$ then also $y \notin$ N. Therefore, $y \in K$ and K is a closed set. ∎

We can now introduce the vector field $\widehat{v}$. Note that the set K we are going to use will be the same as the one in Lemma 4.

Lemma 5 *Let $v = v(y) : S_{++}^{n-1} \mapsto \mathbb{R}^n$ be a vector field that satisfies $(A1) - (A5)$. There exists a continuous inward-pointing vector field $\widehat{v} = \widehat{v}(y) : S_{+}^{n-1} \mapsto \mathbb{R}^n$ that has the same zeros as v.*

Proof Having proved that K is closed, we can now take an open neighbourhood of K, named U, such that $K \subset U \subset \bar{U}$. By this we mean that $U \setminus K$ is not void, in order to assure the continuity of α, and $\bar{U}$ is the union of U and its limits.

Let us now define the vector field that will help us create $\widehat{v}(y)$. Let $v^*(y)$ be as follows:

$$v^*(y) := \frac{1}{y^* \cdot y} y^* - y$$

for

$$y^* = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} ... \frac{1}{\sqrt{n}}).$$

We notice that $\|y^*\|= 1$, which means that y^* belongs to the positive $(n-1)$-dimensional unit sphere. Also, for $y_j = 0$, $v_j^*(y) = \frac{1}{y^* \cdot y} y_j > 0$, thus $v^*(y)$ is an inward pointing vector field.

We then define $\widehat{v}$ as such:

$$\widehat{v}(y) := \alpha(y)v(y) + (1 - \alpha(y))v^*(y).$$

Note that a convex combination of inward pointing vector fields results in an inward pointing vector field, thus $\widehat{v}$ is itself an inward pointing vector field.

Given that we will use the values of α in the following we restate its definition here, for the convenience of the reader:

$$\alpha(y) = \begin{cases} 1, & \text{if } y \in K, \\ \\ 0, & \text{if } y \notin U. \end{cases}$$

Now that we have constructed $\widehat{v}$ we can go on with proving that $\widehat{v}(y) = \mathbf{0}$ if and only if $v(y) = \mathbf{0}$.

$(v = \mathbf{0} \Rightarrow \widehat{v} = \mathbf{0})$ Assume first that $v(y) = \mathbf{0}$. Since all zeros of v are in K, $y \in K \Rightarrow \alpha(y) = 0 \Rightarrow \widehat{v}(y) = 0$. Therefore $v(y) = \mathbf{0} \Rightarrow \widehat{v}(y) = \mathbf{0}$.

$(\widehat{v} = \mathbf{0} \Rightarrow v = \mathbf{0})$ Now assume that $\widehat{v}(y) = \mathbf{0}$. We split the problem into three disjoint cases. By definition, $S_{++}^{n-1} = K \cup N$ and $\partial S_{+}^{n-1} = S_{+}^{n-1} \setminus S_{++}^{n-1}$. Since we are working in S_{+}^{n-1}, we can split the problem by having y in each of K, N and ∂S_{+}^{n-1}. We intend to prove the last two cases by contradiction.

- $(y \in K)$ Let $\widehat{v}(y) = \mathbf{0}$, $y \in K \Rightarrow \alpha(y) = 0 \Rightarrow v(y) = \mathbf{0}$.
- $(y \in N)$ Let $\widehat{v}(y) = \mathbf{0}$ and assume there exists j such that $y \in N_j \Rightarrow v_j > 0$. It follows from the fact that both y and y^* are in S_{+}^{n-1} that $y^* \cdot y \leq 1 \Rightarrow \frac{1}{y^* \cdot y} \geq 1$. Since $y_j^* = \frac{1}{\sqrt{n}}$ and $y_j < \frac{1}{\sqrt{n}}$, we have that $v_j^*(y) = \frac{1}{y^* \cdot y} y_j^* - y_j > 0$.

$\alpha(y) + (1 - \alpha(y)) = 1$, thus we cannot have both $\alpha(y)$ and $1 - \alpha(y)$ equal to zero. Since both $v_j(y)$ and $v_j^*(y)$ are positive, and because at least one of the factors they are multiplied with is positive, we must have that $\widehat{v}_j(y)$ >0. This makes the assumption $\widehat{v}(y) = \mathbf{0}$ a contradiction.

$(y \in \partial S_+^{n-1})$ Let $\widehat{v}(y) = \mathbf{0}$ and $y \in \partial S_+^{n-1}$. This means that $y_j = 0$, for at least one j. $v^*(y)$ is an inward pointing vector field, so $v_j^*(y) > 0$. It is assumed, by convention, that in this case we have $\alpha(y)v(y) = \mathbf{0}$. We therefore obtain $\widehat{v}_j(y) > 0$, leading us to a contradiction of the assumption that $\widehat{v}(y) = \mathbf{0}$.

Therefore, $\widehat{v}(y) = \mathbf{0} \Rightarrow v(y) = \mathbf{0}$.

From the fact that $v(y) = \mathbf{0} \Rightarrow \widehat{v}(y) = \mathbf{0}$ and $\widehat{v}(y) = \mathbf{0} \Rightarrow v(y) = \mathbf{0}$ we conclude that $\widehat{v}(y)$ and $v(y)$ have the same zeros. ∎

Having stated and proved the required lemmas, we can now prove Theorem 3.

Proof of Theorem 3 Let us have a vector field $v : S_{++}^{n-1} \mapsto \mathbb{R}^n$ that satisfies $(A1)-(A5)$. Lemma 2 states the existence of a function ε. Lemma 3 states the existence of a general function α, such that $\alpha(y) := \varepsilon(\beta(y))$. From Lemma 4 we have defined the set K we shall be using, which, in turn, has given us the specific version $\alpha(y)$ required for the proof. By Lemma 5, we can construct a vector field $\widehat{v}(y)$ that has the same zeros as $v(y)$. We use Theorem 2 to prove that there exists a y_0 such that $\widehat{v}(y_0) = \mathbf{0}$. Therefore, there exists a y_0 such that $v(y_0) = \mathbf{0}$, which concludes the proof of Theorem 3. ∎

Chapter 3

Two-Period Economy

We start our observations regarding equilibrium pricing in financial markets in a simple setting, that of two time periods. By that we mean that there is a state zero at time zero, the initial time, and S possible outcomes at time one, the terminal time. There is a finite number I of consumers (or agents) and there is only one good in this economy. Given the existence of a single good, we will be working in the commodity space of $\mathbb{R}^n$, for $n = S + 1$.

We now introduce some definitions in order to properly describe a consumer and the economy. Throughout this section, the superscript ^{i} will represent a random consumer and the subscript $_s$ will represent a random outcome for time 1. To ease notation, we will assume that any statement regarding an investor that contains an ^{i} is applicable to all members of the economy, and $_s$ to all possibilities of time 1. That means, unless specified otherwise, the statement contains "$\forall i = 1, 2, ..., I$" for investors and "$\forall s = 1, 2, ..., S$" for states of the world.

Definition 3 (Wealth Vector) *The wealth of a consumer is defined by a wealth vector that describes his initial endowment* $w^i = (w_0^i, w_1^i, ..., w_S^i)$.

Definition 4 (Utility function) *A utility function is a function* $u^i : \mathbb{R}_+^n \mapsto \mathbb{R}$ *that defines the preference ordering of agent i.*

We now write a number of assumptions on the utility function.

(U'') (Strong Monotonicity)

We assume that the utility function u^i satisfies the following:

(a) $u^i : \mathbb{R}^n_+ \mapsto \mathbb{R}$ is continuous,

(b) u^i is monotone: $\forall x, x^* \in \mathbb{R}^n_+$ with $x \geq x^*, x \neq x^*$, $u^i(x) > u^i(x^*)$.

(U') (Strong Monotonicity and Strict Quasi-concavity)

We assume that the utility function u^i satisfies the following:

(a) $u^i : \mathbb{R}^n_+ \mapsto \mathbb{R}$ is continuous,

(b) u^i is monotone: $\forall x, x^* \in \mathbb{R}^n_+$ with $x \geq x^*, x \neq x^*$, $u^i(x) > u^i(x^*)$,

(c) u^i is quasi-concave: $\forall x, x^* \in \mathbb{R}^n_+$, $0 < \lambda < 1$, $u^i(\lambda x + (1-\lambda)x^*) > \min(u^i(x), u^i(x^*)$.

(U) (Smooth Preferences)

We assume that the utility function u^i satisfies the following:

(a) $u^i : \mathbb{R}^n_+ \mapsto \mathbb{R}$ is continuous on $\mathbb{R}^n_+$ and C^∞ on $\mathbb{R}^n_{++}$,

(b) $U^i(x) = \{x' \in \mathbb{R}^n_+ | u^i(x') \geq u^i(x)\} \subset \mathbb{R}^n_{++}, \forall x \in \mathbb{R}^n_{++}$,

(c) For each $x \in \mathbb{R}^n_{++}, (\frac{\partial u^i(x)}{\partial x_1}, ..., \frac{\partial u^i(x)}{\partial x_n}) \in \mathbb{R}^n_{++}$,

(d) For each $x \in \mathbb{R}^n_{++}, \sum_{j=1}^n \sum_{k=1}^n h_j h_K \frac{\partial^2 u^i(x)}{\partial x_j \partial x_k} < 0 \ \forall h \in \mathbb{R}^n, h \neq 0$, such that $\sum_{j=1}^n h_j \frac{\partial u^i(x)}{\partial x_1} = 0$.

Remark 2 *Let $y = (y_0, y_1, ..., y_S) \in \mathbb{R}^{S+1}$. We write $y_{\mathbf{1}} = (y_1, ..., y_S)$ to represent the vector of components at time 1, such that $y = (y_0, y_{\mathbf{1}})$.*

3.1 Contingent Market Equilibrium

In the current setting, an investor is described by his endowment, which quantifies the amount of wealth he receives in the initial state and in each possible terminal state, and by his utility function, which maps the allocation to a real positive number. Knowing this for each agent, we can fully describe a one-good contingent market.

Definition 5 (Contingent Market Economy) *Let* $U = (u^1, u^2, ..., u^I)$ *and* $W = (w^1, w^2, ..., w^I)$. *Then* $E(U, W)$ *is the associated contingent market economy.*

The financial problem any investor faces is how to maximize utility, given the characteristics of the economy. We consider utility to be a function of consumption (or allocation), and thus we formally define an allocation and the feasible character of it.

Definition 6 (Allocation) *An allocation* $x=(x^1, x^2, ..., x^I)$ *for the economy* $E(U, W)$ *consists of a consumption stream* $x^i \in \mathbb{R}^n$ *for each agent* i.

Definition 7 (Feasible Allocation) *An allocation* x *is feasible if*

1. $x^i \in \mathbb{R}^n_+, \forall i,$
2. $\sum_{i=1}^{I}(x^i - w^i) \leq \mathbf{0}.$

In a contingent market economy we assume the existence of only contingent contracts.

Definition 8 (Contingent Contract) *A contingent contract for state* s *is a promise to deliver one unit of good in state* s *and nothing otherwise.*

The vector of contingent market prices is $\pi = (\pi_0, \pi_1, ..., \pi_S)$, where π_s is the price of the contingent contract at state s. π_s is payable at time zero and it is measured in units of account at time zero.

We assume that there exists a contingent contract for every state s at time one. Thus, the consumer can use his endowment w^i to form his income $\sum_{s=0}^{S} \pi_s w_s^i$ and fund the consumption stream x^i, such that

$$\sum_{s=0}^{S} \pi_s x_s^i \leq \sum_{s=0}^{S} \pi_s w_s^i.$$

Agent i's utility function u^i is monotonic and x^i is assumed to be continuous, therefore we can maximize the consumption up to a level where $\sum_{s=0}^{S} \pi_s x_s^i = \sum_{s=0}^{S} \pi_s w_s^i$ (or $\pi \cdot (x^i - w^i) = 0$ in vector notation) without loss of generality. This enables us to define the budget set:

Definition 9 (Contingent Market Budget Set) *Given a wealth endowment w^i and a pricing vector π, we define the contingent market budget set as*

$$B(\pi, w^i) := \{x^i \in \mathbb{R}_+^n \mid \pi \cdot (x^i - w^i) = 0\}.$$

The following proposition formalizes the intuition that contingent contracts are not free. It is also directly relevant to the proof of the contingent market equilibrium.

Lemma 6 (Absence of Free Contracts on Contingent Markets) *Let the utility functions satisfy (U''), then the following conditions are equivalent:*

1. *The problem max $\{u^i(x^i) \mid x^i \in B(\bar{\pi}, w^i)\}$ has a solution.*
2. *$\bar{\pi} \in \mathbb{R}_{++}^n$.*
3. *$B(\bar{\pi}, w^i)$ is compact.*

Proof We will show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$(1 \Rightarrow 2)$ Assume $\pi_{s_0} = 0$. By the monotonicity of the utility function, consumer i will demand an unbounded quantity of the contingent contract for state s_0, which makes problem 1 have no solution. Thus, $\bar{\pi} \in \mathbb{R}_{++}^n$.

$(2 \Rightarrow 3)$ By the fact that the pricing vector is strictly positive,

$$x_s^i \leq \frac{1}{\bar{\pi}_s} \sum_{s=0}^{S} \bar{\pi}_s x_s^i \leq \frac{1}{\bar{\pi}_s} \sum_{s=0}^{S} \bar{\pi}_s w_s^i.$$

Moreover, consumption is non-negative, so $0 \leq x_s^i$. By that, consumption is bounded, which makes $B(\bar{\pi}, w^i)$ bounded as well. Since the map from x^i to $\bar{\pi} x^i$ is continuous, $B(\bar{\pi}, w^i)$ is closed and, therefore, compact.

$(3 \Rightarrow 1)$ u^i is continuous, therefore it attains its maximum on a compact set, which is $B(\bar{\pi}, w^i)$.

∎

We can now define the concept of contingent market equilibrium. It should be noted that such an equilibrium is the result of trading between agents that behave as stated in the model.

Definition 10 (Contingent Market Equilibrium (CME)) *For the economy $E(U, W)$, a contingent market equilibrium is a pair of allocation and price vector $(\bar{x}, \bar{\pi}) \in \mathbb{R}_+^{nI} \times \mathbb{R}_+^n$ such that:*

1. $\bar{x}^i \in$ *argmax* $\{u^i(x^i) \mid x^i \in B(\bar{\pi}, w^i)\}$,
2. $\sum_{i=1}^{I} (\bar{x}^i - w^i) = \mathbf{0}$.

In order to prove the existence of the CME, one must first consider the demand function, which is a way to interpret allocation as a function of prices.

Definition 11 (Demand Function) *We define the demand function $f^i : \mathbb{R}_{++}^n \mapsto \mathbb{R}^n$ by*

$$f^i(\pi) := \mathit{argmax}\{u^i(x^i) \mid x^i \in B(\pi, w^i)\}.$$

Definition 12 (Aggregate Excess Demand Function) *We define the excess demand function $Z = Z(\pi) : \mathbb{R}^n_{++} \mapsto \mathbb{R}^n$ by*

$$Z(\pi) := \sum_{i=1}^{I} (f^i(\pi) - w^i).$$

For the rest of the section, our purpose will be to prove the existence of the CME. We will first express the CME in terms of the aggregate excess demand function and a vector of demand functions. Then we shall state some properties of the demand function which are directly linked to $(A1) - (A5)$. Based on them, we shall prove that the aggregate excess demand function has some properties that are analogous to $(A1) - (A5)$. Furthermore, we will need to also prove that the aggregate excess demand function represents a vector field on unit sphere. The proof of existence of CME will conclude by the application of Theorem 3 to $Z(\pi)$.

We start by proving that we can express the CME in an equivalent way, in terms of a vector of $f^i(\pi)$ and $Z(\pi)$.

Lemma 7 (Equivalence of CME) *Let (U') hold. $(\bar{x}, \bar{\pi}) \in \mathbb{R}^{nI}_+ \times \mathbb{R}^n_+$ is a CME if and only if*

1. $\bar{x} = (f^1(\bar{\pi}), f^2(\bar{\pi}), ..., f^I(\bar{\pi}))$,

2. $Z(\bar{\pi}) = \mathbf{0}$.

Proof From the strict quasi-concavity of the utility function, there exists only one $\bar{x}^i$ that maximizes utility. Thus

$$\bar{x}^i = \text{argmax}\{u^i(x^i) \mid x^i \in B(\bar{\pi}, w^i)\} = f^i(\bar{\pi}).$$

If we replace it in the expression of $Z(\pi)$, we obtain

$$Z(\bar{\pi}) = \sum_{i=1}^{I} (f^i(\bar{\pi}) - w^i) = \sum_{i=1}^{I} (\bar{x}^i - w^i) = \mathbf{0}.$$

$\blacksquare$

We are now going to enumerate the properties of $f^i(\pi)$ that are relevant to the CME proof of existence and we shall also prove the more accessible ones.

$(A_0 1)$ $f^i(\pi) \geq 0$.

Since $f^i(\pi) = x^i$, and $x \geq 0$ by definition, we must have $f^i(\pi) \geq 0$ for any π.

$(A_0 2)$ f^i is continuous.

A consumer is free to choose any allocation that belongs to the contingent market budget set, and therefore allocation is continuous. By that, the demand function is continuous, as well.

$(A_0 3)$ $\pi \cdot f^i(\pi) = \pi \cdot w^i$.

Since, at any given level, $f^i(\pi) = x^i$, and we have the budget set condition $\pi \cdot x^i = \pi \cdot w^i$, we conclude by replacing x^i with $f^i(\pi)$.

$(A_0 4)$ $f^i(\alpha\pi) = f^i(\pi)$, for α a positive constant.

For the price vector π we have $\pi \cdot f^i(\pi) = \pi \cdot w^i$; for the price vector $\alpha\pi$ we have $(\alpha\pi) \cdot f^i(\alpha\pi) = (\alpha\pi) \cdot w^i$. If the second equation is divided by α and if it is combined (by subtraction) with the first equation, we end up with $\pi \cdot (f^i(\pi) - f^i(\alpha\pi)) = 0$. All entries of π are strictly positive, which means that we cannot have this equation unless $f^i(\alpha\pi) = f^i(\pi)$.

$(A_0 5)$ If $\pi^\alpha \in \mathbb{R}^n_{++}$ is such that $\pi^\alpha \to \partial\mathbb{R}^n_{++}$ and if $\pi w^i > 0$, then $\| f^i(\pi^\alpha) \| \to \infty$ as $\alpha \to \infty$.

As prices become very small, the demand for the goods becomes very large. In the limiting case, the demand becomes arbitrarily large.

Let us now formally prove the properties of Z.

Lemma 8 (Properties of Z) *Z satisfies $(A1) - (A5)$.*

Proof $(A1)$ $Z(\pi) \geq -\sum_{i=1}^{I} w^i$.

$Z(\pi)$ can be written as $\sum_{i=1}^{I} f^i(\pi)$-$\sum_{i=1}^{I} w^i$. f^i is lower bounded by 0, thus Z is lower bounded by $-\sum_{i=1}^{I} w^i$.

$(A2)$ Z is continuous.

The functions f^i are continuous for all $i = 1, ..., n$. Z is a linear combination of f^i's and constants w^i, thus Z is continuous.

$(A3)$ $\pi \cdot Z(\pi) = 0$.

$$\pi \cdot Z(\pi) = \pi \cdot \sum_{i=1}^{I}(f^i(\pi) - w^i) = \sum_{i=1}^{I}(\pi \cdot f^i(\pi)) - \sum_{i=1}^{I}(\pi \cdot w^i) =$$

$$\sum_{i=1}^{I}(\pi \cdot w^i) - \sum_{i=1}^{I}(\pi \cdot w^i) = 0.$$

$(A4)$ $Z(\alpha\pi) = Z(\pi)$, for α a positive constant.

$$Z(\pi) = \sum_{i=1}^{I}(f^i(\pi) - w^i) = \sum_{i=1}^{I}(f^i(\alpha\pi) - w^i) = Z(\alpha\pi).$$

$(A5)$ If $\pi^\alpha \in \mathbb{R}^n_{++}$ such that $\pi^\alpha \to \pi \in \partial\mathbb{R}^n_{++}$, $\pi \neq 0$, then $\|Z(\pi^\alpha)\| \to \infty$ as $\alpha \to \infty$.

The reasoning is similar to the one for f^i, as prices tend to zero the demand for them tends to infinity. In particular, if $\pi^\alpha \to \partial\mathbb{R}^n_{++}$, then there exists at least one i such that $\|f^i(\pi^\alpha)\| \to \infty$ as $\alpha \to \infty$, and $Z(\pi^\alpha)$ has the same boundary behaviour.

∎

Lemma 9 (Vector field on unit sphere) *The aggregate demand function $Z(\pi)$ can be described as a vector field on the unit sphere, that is $Z : S^{n-1}_{++} \mapsto \mathbb{R}^n$.*

Proof By the virtue of the fact that the aggregate excess demand varies only when the **relative** prices change, given by $(A3)$, we can impose any normalization on the prices. If we have the original price vector $\pi' = (\pi'_0, \pi'_1, ..., \pi'_{n-1})$ (remember that $n = S + 1$), let us denote α by

$$\alpha = \sqrt{\sum_{i=1}^{n} (\pi'_i)^2}.$$

By Lemma 6 the prices π'_i are positive, thus α is positive. This enables us to define a new price vector, π, by having

$$\pi = \frac{\pi'}{\alpha} = (\frac{\pi'_0}{\alpha}, ..., \frac{\pi'_n}{\alpha}).$$

The dot product of π and itself is trivially 1, that is

$$\pi^2 = \frac{1}{\alpha^2} \sum_{i=1}^{n} (\pi'_i)^2 = 1,$$

which is the equation of an $(n-1)$-dimensional unit sphere.

Let $g(\pi) = \sum_{i=1}^{n} (\pi_i^2)$, such that $g(\pi) = 1$ is the equation of the unit sphere, and let $\widehat{\pi} \in S^{n-1}$ be a point on the unit sphere. We then take the tangent hyperplane of $g(\pi)$ at $\widehat{\pi}$, given by

$$\nabla g(\widehat{\pi}) \cdot (\pi - \widehat{\pi}) = 0. \tag{3.1}$$

The gradient of $g(\widehat{\pi})$ is the vector of partial derivatives with respect to each component $\widehat{\pi}_i$, and thus

$$\nabla g(\widehat{\pi}) = (\partial \sum_{i=1}^{n} (\pi_i^2) / \partial \widehat{\pi}_1, ..., \partial \sum_{i=1}^{n} (\pi_i^2) / \partial \widehat{\pi}_n) = 2\widehat{\pi}.$$

Equation (3.1) becomes $2\widehat{\pi} \cdot (\pi - \widehat{\pi}) = 0$, which is equivalent to $\widehat{\pi} \cdot \pi = \widehat{\pi}^2$. $\widehat{\pi} \in S^{n-1}$, which means that $\widehat{\pi}^2 = 1$. Therefore, (3.1) is equivalent to

$$\widehat{\pi} \cdot \pi = 1.$$

If we translate this hyperplane by one unit towards the origin, we obtain the tangent space at $\widehat{\pi}$

$$T_{\widehat{\pi}} S^{n-1} = \{z \in \mathbb{R}^n \mid \widehat{\pi} \cdot z = 0\}.$$

By the fact that $\pi \cdot Z(\pi) = 0$, $Z(\pi)$ belongs to the tangent space $T_{\pi} S^{n-1}$ and therefore it is a vector field on the unit sphere. ∎

Theorem 4 (Existence of CME) *Let (U') hold. If $\sum_{i=1}^{I} w^i \in \mathbb{R}^n_{++}$, then the contingent economy $E(U, W)$ has a CME.*

Proof We have shown that the excess demand function $Z(\pi) : S^{n-1}_{++} \mapsto \mathbb{R}^n$ satisfies $(A1) - (A5)$. Therefore, the existence of a zero for the map $\pi \mapsto Z(\pi)$ follows from Theorem 3. ∎

3.2 Financial Market Equilibrium

We now expand our economy by considering a broader type of contracts, which would be general financial contracts (or securities). We start from the contingent market economy $E(U, W)$ and we endow it with a matrix of financial contracts V.

Definition 13 (Financial Contract) *A financial contract j is a promise to deliver $V_s^j \in \mathbb{R}$ units of the good (or income) in state s. Its price q_j, payable at date 0, is measured in the unit of account at date 0.*

Definition 14 (Security Payoff Matrix) *The $S \times J$ matrix V represents the financial contracts in the economy, where*

$$V = \begin{bmatrix} V_1^1 & \cdots & V_1^J \\ \vdots & \ddots & \vdots \\ V_S^1 & \cdots & V_S^J \end{bmatrix}.$$

Let $q = (q_1, ..., q_J)$ denote the vector of security prices.

Then the matrix of security prices is

$$P(q, V) = \begin{bmatrix} -q \\ V \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_I \\ V_1^1 & \cdots & V_1^J \\ \vdots & \ddots & \vdots \\ V_S^1 & \cdots & V_S^J \end{bmatrix}.$$

Definition 15 (Financial Economy) *Let $U = (u^1, u^2, ..., u^I)$, $W = (w^1, w^2, ..., w^I)$ and V is the matrix of financial contracts. Then $E(U, W, V)$ is the associated financial economy.*

An efficient model of a financial economy does not admit arbitrage opportunities. Given (q, V) there are no arbitrage opportunities in the financial market if there does not exist a portfolio which generates a semi-positive income stream. That is, there does not exist $z \in \mathbb{R}^J$ such that $Pz > 0$. q is a vector of no-arbitrage security prices if given (q, V) there are no arbitrage opportunities on the financial market.

Let us define the financial market budget set.

Definition 16 (Financial Market Budget Set) *Let $z^i = (z^i_1, ..., z^i_J) \in \mathbb{R}^J$ denote the i'th agent's portfolio giving the number of units the agent bought ($z^i_j > 0$) or sold ($z^i_j < 0$).*

We then define the financial market budget set as

$$\mathbb{B}(q, w^i, V) := \{x^i \in \mathbb{R}^n_+ \mid x^i - w^i = Wz^i, z^i \in \mathbb{R}^J\}.$$

We shall use the notation $(x^i; z^i)$ to represent that the agent chooses the portfolio z^i such that $x^i - w^i = Wz^i$, also said that z^i finances x^i. At optimality, $(\bar{x}^i; \bar{z}^i)$ symbolizes the fact that $\bar{x}^i$ is one of the allocations which maximizes utility and z^i finances x^i.

Definition 17 (Financial Market Equilibrium (FME)) *A financial market equilibrium for the finance economy $E(U, W, V)$ is a pair consisting of actions and prices $((\bar{x}, \bar{z}), \bar{q}) \in \mathbb{R}^{nI}_+ \times \mathbb{R}^{JI} \times \mathbb{R}^J$ such that:*

1. $(\bar{x}^i; \bar{z}^i) \in$ *argmax* $\{u^i(x^i) \mid (x^i; z^i) \in \mathbb{B}(\bar{q}, w^i, V)\}$
2. $\sum_{i=1}^{I} \bar{z}^i = \boldsymbol{0}$

In order to prove the existence of an FME, we must first prove that it can be written in an equivalent way, through the normalized equilibrium.

Definition 18 (Normalized Equilibrium) *We define the normalized budget set as*

$$\boldsymbol{B}(\pi, w^i, V) := \{x^i \in \mathbb{R}^n_+ \mid \pi \cdot (x^i - w^i) = 0, x^i_1 - w^i_1 \in \langle V \rangle\}.$$

A normalized equilibrium (NE) for finance economy $E(U, W, V)$ is a pair consisting of an allocation and a vector of prices $(\bar{x}, \bar{\pi}) \in \mathbb{R}^{nI}_{+} \times \mathbb{R}^{n}_{+}$ such that:

1. $\bar{x}^1 \in argmax\ \{u^1(x^1) \mid x^1 \in B(\bar{\pi}, w^1)\}$,

   $\bar{x}^i \in argmax\ \{u^i(x^i) \mid x^i \in \boldsymbol{B}(\pi, w^i, V)\}$,

2. $\sum_{i=1}^{I}(\bar{x}^i - w^i) = \boldsymbol{0}$.

Lemma 10 (Equivalence of FME and NE) *Let $E(U, W, V)$ be a one-good economy satisfying (U).*

1. *If $((\bar{x}, \bar{z}), \bar{q})$ is an FME and if $\bar{\pi}^1$ is agent 1's present value vector, then $(\bar{x}, \bar{\pi}^1)$ is an NE.*

2. *If $(\bar{x}, \bar{\pi})$ is an NE with $\bar{\pi} = (1, \bar{\pi}_{\boldsymbol{1}})$, then there exist portfolios $\bar{z} = (\bar{z}^1, \bar{z}^2, ..., \bar{z}^I)$ and security prices $\bar{q} = \bar{\pi}_{\boldsymbol{1}} V$ such that $((\bar{x}, \bar{z}), \bar{q})$ is a FME.*

Proof The proof goes beyond the scope of this project, but can be found in most textbooks about equilibria in incomplete markets, such as Proposition 10.3 in [3].

∎

We shall now prove the existence of a FME throughout the rest of the section. It will be similar to the proof of the CME in the contingent market section, the only fundamental difference will be that we will first prove the existence of a NE and then use Lemma 10 to prove the existence of the financial market equilibrium.

Definition 19 (Financial Demand Functions) *Define the demand functions f^1 and f^i_V as:*

$$f^1 = f^1(\pi) : \mathbb{R}^n_{++} \mapsto \mathbb{R}^n,\ f^1(\pi) := argmax\{u^1(x^1) | x^1 \in B(\pi, w^1)\},$$

$$f^i_V = f^i_V(\pi) : \mathbb{R}^n_{++} \mapsto \mathbb{R}^n,\ f^i_V(\pi) := argmax\{u^i(x^i) | x^i \in \boldsymbol{B}(\pi, w^i, V)\}.$$

We note that $f^1(\pi)$ is exactly the demand function from a contingent market (for $i = 1$), meaning that it has all the properties of the $f^i(\pi)$ from the contingent market. As such, $f^1(\pi)$ satisfies $(A_0 1) - (A_0 5)$. f_V^i satisfies $(A_0 1) - (A_0 4)$, but does not have a similar boundary behaviour.

Definition 20 (Financial Aggregate Excess Demand Function) *We define the financial aggregate excess demand function* $Z_V : \mathbb{R}^n_{++} \mapsto \mathbb{R}^n$ *as*

$$Z_V(\pi) := f^1(\pi) - w^1 + \sum_{i=2}^{I} (f_V^i(\pi) - w^i).$$

Lemma 11 (Equivalence of NE) $(\bar{x}, \bar{\pi}) \in \mathbb{R}^{nI}_+ \times \mathbb{R}^n_+$ *is a normalized equilibrium if and only if*

1. $\bar{x} = (f^1(\bar{\pi}), f_V^i(\bar{\pi}), ..., f_V^i(\bar{\pi}))$,
2. $Z_V(\bar{\pi}) = \boldsymbol{0}$.

Proof At optimality,

$$\bar{x}^1 = f^1(\bar{\pi}) \text{ and } \bar{x}^i = f_V^i(\bar{\pi}), i = 2, ..., n.$$

If we replace in $Z_V(\bar{\pi})$ we obtain

$$Z_V(\bar{\pi}) = f^1(\bar{\pi}) - w^1 + \sum_{i=2}^{I} (f_V^i(\bar{\pi}) - w^i) = \sum_{i=1}^{I} (\bar{x}^i - w^i) = \mathbf{0}.$$

∎

The following lemma states that $Z_V(\pi)$ satisfies $(A1) - (A5)$, which will enable us to use Theorem 3 in the proof of existence of FME. Fundamentally, the proof of Lemma 12 is going to be the same as the one presented in Section 1. The argument that $Z_V(\pi)$ is a vector field on the unit sphere is completely analogous to the one about $Z(\pi)$ in the setting of a contingent market, so it shall be omitted.

Lemma 12 (Properties of Z_V) *$Z_V(\pi)$ satisfies $(A1) - (A5)$.*

Proof $(A1)$ $Z_V(\pi) \geq -\sum_{i=1}^{I} w^i$.

Since both $f^1(\pi)$ and $f_V^i(\pi)$ are lower bounded by 0, we have that

$$Z_V(\pi) \geq -w^1 + \sum_{i=2}^{I}(-w^i) = -\sum_{i=1}^{I} w^i.$$

$(A2)$ Z_V is continuous.

Z_V is a linear combination of f^1, f_V^i's and constants w^i, thus Z_V is continuous by the fact that both f^1 and f_V^i are continuous.

$(A3)$ $\pi \cdot Z_V(\pi) = 0$.

We know that $\pi \cdot f^1(\pi) = \pi \cdot w^1$ and $\pi \cdot f_V^i(\pi) = \pi \cdot w^i$. Therefore

$$\pi \cdot Z_V(\pi) = \pi \cdot f^1(\pi) - \pi \cdot w^1 + \sum_{i=2}^{I}(\pi \cdot f_V^i(\pi) - \pi \cdot w^i) = 0.$$

$(A4)$ $Z_V(\alpha\pi) = Z_V(\pi)$, for α a positive constant.

$$Z_V(\alpha\pi) = f^1(\alpha\pi) - w^1 + \sum_{i=2}^{I}(f_V^i(\alpha\pi) - w^i) = f^1(\pi) - w^1 + \sum_{i=2}^{I}(f_V^i(\pi) - w^i) = Z_V(\pi).$$

$(A5)$ If $\pi^\alpha \in \mathbb{R}^n_{++}$ such that $\pi^\alpha \to \pi \in \partial\mathbb{R}^n_{++}$, $\pi \neq 0$, then $\|Z_V(\pi^\alpha)\| \to \infty$ as $\alpha \to \infty$.

The boundary behaviour is inherited from f^1 and it quantifies the intuition that, as prices fall to nothing, demand increases unboundedly.

∎

Theorem 5 (Existence of FME) *Let $E(U, W, V)$ be a one-good economy satisfying (U). If $w^i \in \mathbb{R}^N_{++}$, then there exists a financial equilibrium $((\bar{x}, \bar{z}), \bar{q})$.*

Proof We have shown that $Z_V(\pi)$ satisfies $(A1) - (A5)$. Therefore, by Theorem 3, $Z_V(\pi)$ has a zero, which means that a financial economy has a normalized equilibrium. By Lemma 10, the normalized equilibrium is equivalent to a financial equilibrium, which concludes the proof of existence of a financial equilibrium. ∎

Remark 3 *Assumption* (U') *is sufficient for the financial market to have a normalized equilibrium.*

3.3 Completeness

An important aspect to take into account when discussing a financial market is whether it is complete or not. In a complete market an investor can replicate any financial security through a portfolio of securities, which allows for direct pricing through replication.

Recall the payoff matrix

$$P(q, V) = \begin{bmatrix} -q \\ V \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_I \\ V_1^1 & \cdots & V_1^J \\ \vdots & \ddots & \vdots \\ V_S^1 & \cdots & V_S^J \end{bmatrix}.$$

One can decompose the space of all possible trades $\mathbb{R}^{S+1}$ into $\langle P \rangle$ and the orthogonal complement $\langle P^\perp \rangle$. As such, we have that $\dim\langle P \rangle + \dim\langle P^\perp \rangle = S + 1$.

Definition 21 (Completeness) *Let the market subspace $\langle P \rangle$ be arbitrage free. If the market subspace has maximal dimension then the financial market is said to be complete. If it has less than maximal dimension then the market is incomplete.*

Proposition 1 (Characterization of Completeness) *In a two period finance economy $E(U, W, V)$, the financial markets are complete for any vector of no-arbitrage security prices q if and only if $rankV = S$.*

Proof Let q be a vector of no-arbitrage security prices. We then must have that there exists a vector of prices $\pi \in \mathbb{R}^{S+1}$ such that $q = \sum_{s=1}^{S} \pi_s V_s$. This means that q is a linear combination of the rows of V. Therefore, $rankP = rankV = S$. ∎

We are now aiming to prove a property that characterizes incomplete markets. We are going to look at an incomplete financial market at equilibrium. In such a case, agents will have distinct present-value vectors, with the exception of a set of measure zero (concept introduced below). In order to state the theorem we must first introduce the notions of cube and full measure.

Let $a, b \in \mathbb{R}^n$ with $a_i < b_i$ for all $i = 1, ..., n$. The cube $C(a, b)$ generated by (a, b) is defined as

$$C(a, b) = \{x \in \mathbb{R}^N | a_i < x_i < b_i, \forall i\}.$$

The measure $\alpha(C(a, b))$ is defined as

$$\alpha(C(a, b)) = \prod_{i=1}^{n} (b_i - a_i).$$

A subset $S \subset \mathbb{R}^n$ is of measure zero if for all $\varepsilon > 0$ there exists a countable collection of cubes $(C^i_\varepsilon)_{i=1}^\infty$ such that

$$S \subset \bigcup_{i=1}^{\infty} C^i_\varepsilon \ \ \textit{and} \ \ \sum_{i=1}^{\infty} \alpha(C^i_\varepsilon) \leq \varepsilon.$$

A set has full measure if its complement has measure zero.

We will also need to define what a present-value vector is.

Definition 22 (Present-Value Vector) *We define the normalized gradient of agent i at $\bar{x}^i$ as*

$$\lambda^i_0(\bar{x}^i) := \frac{\partial u^i(\bar{x}^i)}{\partial x^i_0}.$$

Then we define the present-value vector of agent i at $\bar{x}^i$ as

$$\pi^i(\bar{x}^i) := \frac{\nabla u^i(\bar{x}^i)}{\lambda^i_0(\bar{x}^i)}.$$

We note that the components of the vector are the rates of substitution between the state s allocation and consumption at the initial time, that is

$$\pi^i_s(\bar{x}^i) = \frac{\partial u^i(\bar{x}^i)}{\partial x^i_s} / \frac{\partial u^i(\bar{x}^i)}{\partial x^i_0}.$$

Theorem 6 *Let $E(U, W, V)$ be a family of economies parametrized by the agents' endowments $w \in \Omega$, in which*

1. *there are at least two agents ($I \geq 2$),*
2. *the vector of utility functions $u = (u^1, ..., u^I)$ satisfies (U),*
3. *the financial markets are incomplete (rank$V = J < S$).*

Then there exists a set of full measure $\Omega^* \subset \Omega$ such that, if $w \in \Omega^*$, the present-value vectors of all agents are distinct in each of the financial market equilibria $((\bar{x}, \bar{z}), \bar{q})$, i.e

$$\pi^i(\bar{x}^i) \neq \pi^j(\bar{x}^j), \ \forall i \neq j.$$

Proof The solution involves solving systems of parametric equations and is not given here. One could find it as the proof of Theorem 11.6 in [3]. ∎

Thus, one significant drawback of an incomplete setting is the fact that agents assign different value to the marginal utility offered by consumption, which leads to unequal marginal rates of substitution.

On the other hand, in a complete financial market, not only are the present value vectors equal for the terminal time (theorem stated formally in the next section), but the financial market is equivalent to the contingent market.

Theorem 7 (FM-CM Equivalence with Complete Markets) *Let $E(U, V, W)$ be a one-good economy satisfying (U). Let the financial market be complete, i.e $\langle V \rangle = \mathbb{R}^S$. Then we have the following:*

1. *If $((\bar{x}, \bar{z}), \bar{q}$ is a FME, then $(\bar{x}, \bar{\pi})$ is a CME where $\bar{\pi} = (1, \bar{\pi}_{\mathbf{1}}) \in \mathbb{R}^{S+1}_{++}$ is the unique vector of state prices that satisfies $\bar{q} = \bar{\pi}_{\mathbf{1}} V$.*
2. *If $(\bar{x}, \bar{\pi})$ is a CME with $\bar{\pi} = (1, \bar{\pi}_{\mathbf{1}}) \in \mathbb{R}^{S+1}_{++}$ then there exists a vector of portfolios $\bar{z} = (\bar{z}^1, \bar{z}^2, ..., \bar{z}^I)$ and a vector of security prices $\bar{q} = \bar{\pi}_{\mathbf{1}} V$ such that $((\bar{x}, \bar{z}), \bar{q})$ is a FME.*

Proof If the market is complete, then the span of the payoff matrix covers the entire space of commodities, i.e

$$\langle V \rangle = \mathbb{R}^S \Longleftrightarrow \mathbf{B}(\pi, w^i, V) = B(\pi, w^i),$$

which means that the CME is equivalent to the NE.

Lemma 10 states that the NE is equivalent to the FME, which concludes the proof. ∎

3.4 Efficiency

An important aspect when considering a financial economy is efficiency. There are numerous perspectives through which we can observe different kinds of optimality, but let us consider Pareto optimality in different circumstances.

Definition 23 (Pareto Optimality) *An allocation $\bar{x} = (\bar{x}^1, \bar{x}^2, ..., \bar{x}^I)$ is a Pareto optimum if*

1. *$\bar{x}$ is feasible ($\bar{x} \in F$),*
2. *$\nexists x = (x^1, x^2, ..., x^I) \in F$ such that $u^i(x^i) \geq u^i(\bar{x}^i) \forall i$ and $\exists i$ such that $u^i(x^i) > u^i(\bar{x}^i)$.*

In other words, an allocation is Pareto inefficient if there is at least an investor that can improve his utility without worsening anyone else's. This gives us a powerful, even if simple, standard on which we can compare allocations.

Let us now consider a contingent economy and its equilibrium.

Theorem 8 (Pareto Optimality of CME) *Let $E(U, W)$ be an economy satisfying (U''). If $(\bar{x}, \bar{\pi})$ is a contingent market equilibrium, then the allocation $\bar{x}$ is a Pareto Optimum.*

Proof Let us prove this statement by contradiction. Let us assume that $\bar{x}$ is Pareto inefficient and there exists $x \in F$ such that $u^i(x^i) \geq u^i(\bar{x}^i)$ and there exists a j such that $u^j(x^j) > u^j(\bar{x}^j)$. By the monotonicity of the utility function and by the fact that $\bar{x}$ optimizes utility on the budget set $B(\bar{\pi}, w^j)$, we have $\bar{\pi}x^i \geq \bar{\pi}w^i$ for all $i \neq j$ and $\bar{\pi}\bar{x}^j > \bar{\pi}w^j$. If we sum up, we obtain

$$\bar{\pi} \sum_{i=1}^{I} (x^i - w^i) > 0.$$

$\bar{\pi}$ contains strictly positive entries, which means that $\sum_{i=1}^{I}(x^i - w^i) > 0$. This violates the feasibility assumption, and we obtain the required contradiction. ∎

Lemma 13 *Let $E(U, W)$ be an economy satisfying (U). An allocation $\bar{x} = (\bar{x}^1, \bar{x}^2, ..., \bar{x}^I)$ is Pareto optimal if and only if*

1. *$\bar{x} \in F$*

2. *$\exists \bar{\pi}_1 \in \mathbb{R}^S$ such that*

$$\pi_1^1(\bar{x}^1) = \pi_1^2(\bar{x}^2) = ... = \pi_1^I(\bar{x}^I) = \bar{\pi}_1.$$

Proof Such a proof can be found in [3], Section 7. ∎

Theorem 9 (Inefficiency of Incomplete Markets) *Let $E(U, W, V)$ be an economy satisfying:*

1. *there are at least two agents ($I \geq 2$),*

2. *the vector of utility functions $u = (u^1, ..., u^I)$ satisfies (U),*

3. *the financial market is incomplete (rank $V = J < S$).*

Then there exists a set of full measure $\Omega* \subset \Omega$, such that if $\omega \subset \Omega^*$ then for each financial market equilibrium $((\bar{x}, \bar{z}), \bar{q}))$ of the economy $E(U, W, V)$ the allocation $\bar{x}$ is Pareto inefficient.

Proof From Theorem 6 we have that the vectors of present values are different. Lemma 13 gives us the fact that an allocation is Pareto optimal only when the present-value vectors are equal, therefore the vector of allocations is Pareto inefficient in an incomplete financial market. ∎

Chapter 4

Multi-Period Economy

The first step towards a more realistic model is to consider several periods of time. Let us extend the number of time periods to T, a finite natural number greater than two. We can, thus, model a gradual inflow of information, where an investor knows a bit more after every period (or, at least, the same). This will allow us to consider more complex types of financial derivatives, such as financial contracts with dividends, where agents must be aware of the future capital value of the security. We note that this chapter is aimed at only presenting the setting of a multi-period model and we shall be focusing on the difference between this framework and the two-period setting. New results will not be introduced here, but we shall present the ones which are analogous to the two-period model.

To formalize the concept of partial information, we introduce the set $\mathbf{S} = \{1, ..., S\}$ with finite cardinality S, which contains all possible states, and we assume that a state $s \in \mathbf{S}$ is selected at random.

Definition 24 (Partition) *A partition of* $\boldsymbol{S}$ *is a collection of mutually disjoint subsets of* $\boldsymbol{S}$ *whose union is* $\boldsymbol{S}$.

We then have a sequence of partitions of $\mathbf{S}$, $\mathbb{F} = (\mathbb{F}_0, \mathbb{F}_1, ..., \mathbb{F}_T)$, such that $\mathbb{F}_0 = \mathbf{S}$, $\mathbb{F}_T = \{\{1\}, ..., \{S\}\}$ and $\mathbb{F}_t$ is finer than $\mathbb{F}_{t-1} \forall t = 1, ..., T$. A partition $\mathbb{F}_t$ is finer than

$\mathbb{F}_{t-1}$ if

$$\sigma \in \mathbb{F}_t,\ \sigma' \in \mathbb{F}_{t-1}\ \Rightarrow\ \sigma \subseteq \sigma' \text{ or } \sigma \cap \sigma' = \oslash.$$

This is how we can quantify the gradual inflow of information, starting from no information at time 0 and ending with full information at time T. We now extend this perspective within the framework of the event-tree approach, which regards each partition $\mathbb{F}_t$ as a node of a tree. We shall now give formal definitions of an event-tree and subtree and of a predecessor and successor node.

Definition 25 (Event-tree) *Let $\mathbb{F} = (\mathbb{F}_0, ..., \mathbb{F}_T)$ be a sequence of partitions of $\boldsymbol{S}$, with $\mathbb{F}_0 = \boldsymbol{S}$, $\mathbb{F}_T = \{\{1\}, ..., \{\boldsymbol{S}\}\}$ and $\mathbb{F}_t$ finer than $\mathbb{F}_{t-1}$. For each date $t \in \boldsymbol{T}$ and each subset $\sigma \in \mathbb{F}_t$ the pair (t, σ) is called a date-event of node. The set $\mathbb{D}$ consisting of all date-events is called the event-tree induced by $\mathbb{F}$, that is*

$$\mathbb{D} := \bigcup_{t \in \boldsymbol{T}\ \sigma \in \mathbb{F}_t} (t, \sigma).$$

We shall further use $\xi = (t, \sigma)$ to describe the nodes of $\mathbb{D}$ and $n = \#\mathbb{D}$ as the number of nodes in $\mathbb{D}$.

Definition 26 (Initial Node) *The node $\xi_0 = (0, \boldsymbol{S})$ is called the initial node.*

The set of non-initial nodes is written as $\mathbb{D}^+$, where

$$\mathbb{D}^+ := \mathbb{D} \setminus \xi_0.$$

Definition 27 (Terminal Node) *A node (T, σ) with $\sigma \in \mathbb{F}_T$ is called a terminal node.*

The set of non-terminal nodes is written as $\mathbb{D}^-$, where

$$\mathbb{D}^- := \mathbb{D} \setminus \mathbb{D}_T.$$

Definition 28 (Predecessor) *For each $\xi \in \mathbb{D}^+$, $\xi = (t, \sigma)$, there exists a unique subset $\sigma' \in \mathbb{F}_{t-1}$ such that $\sigma \subset \sigma'$. The node $\xi^- = (t-1, \sigma')$ is called the predecessor of ξ.*

Definition 29 (Successor) *For each $\xi \in \mathbb{D}^-$, $\xi = (t, \sigma)$, the set*

$$\xi^+ = \{\xi' \in \mathbb{D} \mid \xi' = (t+1, \sigma'),\ \sigma' \subset \sigma\}$$

is the set of immediate successors of ξ.

ξ' is a successor of ξ if $\xi' = (t', \sigma')$, $\xi = (t, \sigma)$ satisfy $t' \geq t$, $\sigma' \subset \sigma$. We denote this by $\xi' \geq \xi$.

In the above definition, if we have $t' > t$ we say that ξ' is a strict successor of ξ, written as $\xi' > \xi$.

Definition 30 (Subtree) *For any node $\xi' \in \mathbb{D}$, the set of all successors of ξ' is called the subtree of ξ' (or the subtree that starts at ξ')) and is written as $\mathbb{D}(\xi')$, where*

$$\mathbb{D}(\xi') := \{\xi \in \mathbb{D} \mid \xi \geq \xi'\}.$$

The set of strict successors of ξ' is denoted as $\mathbb{D}^+(\xi')$, where

$$\mathbb{D}^+(\xi') := \{\xi \in \mathbb{D}(\xi') \mid \xi > \xi'\}$$

and the set of non-terminal successors of ξ' is written as $\mathbb{D}^-(\xi')$, for

$$\mathbb{D}^-(\xi') := \{\xi \in \mathbb{D}(\xi') \mid \xi \in \mathbb{D}^-\}.$$

As one can notice, the two-period model is a particularization of such an event tree, where there are two periods of time, with only an initial node and terminal nodes.

A significant change brought up by the advance to the multi-period setting is the way the endowment process is defined. It has to be taken into account the fact that the endowment process defines a function of the nodes in the event-tree that varies over time. We formalize this by the additional index of the states of nature, as such:

$$w^i = (w^i(t,s),\ (t,s) \in \mathbf{T} \times \mathbf{S}).$$

Remark 4 *We will be working under the assumption that the information structure* $\mathbb{F} = (\mathbb{F}_t)_{t \in \boldsymbol{T}}$ *is the same throughout the economy and all agents have access to it.*

Under this assumption, the consistency of the endowment process must be formalized with respect to the restrictions on information, which leads to the following equation that must be satisfied by w^i:

$$w^i(t,s) = w^i(t,s'),\ \forall s, s' \in \sigma \in \mathbb{F}_t,\ \forall t \in [0,\tau],\ \forall \tau \in \mathbf{T}.$$

By this we mean that an agent has as much information about the state s as can be derived by observing the path of the wealth endowment throughout the event-tree.

We can now give a definition of the multi-period commodity space we are going to work under.

Definition 31 (Commodity Space) *Let* $\boldsymbol{S} = \{1, ..., S\}$ *be a set of states and let* $\boldsymbol{T} = \{0, 1, ..., T\}$ *be a set of time periods. Define* $\mathbb{F} = (\mathbb{F}_t)_{t \in \boldsymbol{T}}$ *to be the sequence of partitions which gives the unfolding of information, such that* $\mathbb{F}_0 = \boldsymbol{S}$, $\mathbb{F}_T = \{\{1\}, ..., \{S\}\}$ *and* $\mathbb{F}_t$ *is finer than* $\mathbb{F}_{t-1}$, *for all* $t = 1, .., T$. *The commodity space under* $\boldsymbol{T}, \boldsymbol{S}$ *and* $\mathbb{F}$ *is the vector space*

$$\mathbb{R}^n = \{x = (x(\xi), \xi \in \mathbb{D}) \mid x(\xi) \in \mathbb{R}, \forall \xi \in \mathbb{D}\}.$$

Recall that n is the number of nodes in $\mathbb{D}$. We can observe that the dimension of the commodity space is strictly greater than in the two-time period model. Calculations become more difficult in such an environment because of the added stochasticity brought by an unknown number of 'intermediate' nodes (non-initial, non-terminal).

Let us consider the contingent market.

Definition 32 (Contingent Contract) *A contingent contract for node $\xi \in \mathbb{D}$ is a promise to deliver one unit of the single good at node ξ and nothing otherwise. Its price $\pi(\xi)$, payable at date 0, is measured in units of account at date 0.*

The contingent market budget set and equilibrium definitions are the same as in the two-period, with n being the number of nodes in the event-tree. We restate the definitions, for convenience.

(Contingent Market Budget Set) Given a wealth endowment w^i and a pricing vector π, we define the contingent market budget set as

$$B(\pi, w^i) := \{x^i = (x^i(\xi), \xi \in \mathbb{D}^+) \in \mathbb{R}^n_+ \mid \pi \cdot (x^i - w^i) = 0\}.$$

(Contingent Market Equilibrium (CME)) For the economy $E(\mathbb{D}, U, W)$, a contingent market equilibrium is a pair of allocation and vector prices $(\bar{x}, \bar{\pi}) \in \mathbb{R}^{nI}_+ \times \mathbb{R}^n_+$ such that:

1. $\bar{x}^i \in \text{argmax } \{u^i(x^i) \mid x^i \in B(\bar{\pi}, w^i)\}$
2. $\displaystyle\sum_{i=1}^{I}(\bar{x}^i - w^i) = \mathbf{0}$

Theorem 10 (Existence of CME) *Let (U') hold. If $\sum_{i=1}^{I} w^i \in \mathbb{R}^n_{++}$, then $E(\mathbb{D}, U, W)$ has a contingent market equilibrium.*

In the case of the contingent market, one can link each contingent contract at node ξ to its predecessor ξ^- in terms of price. Consequently, the set of all contingent contracts can be translated in this matter to the initial node ξ_0. This means that all decisions are taken at the initial time. Accordingly, there is no structural difference here between two-period and multi-period and the proof of Theorem 10 is fundamentally the same as for Theorem 4.

Theorem 11 (Pareto Optimality of CME) *Let (U'') hold. Then the CME of economy $E(\mathbb{D}, U, W)$ has a Pareto optimal allocation.*

Similarly to the argument presented for Theorem 10, Pareto optimality comes as a consequence of the fact that all contingent claims are traded at the initial node.

We further expand our research into the financial market context by modelling securities of a higher complexity than commodities. Note that, in this multi-period model, a financial contract can be issued at any non-terminal time $0 \leq t < T$ and it can expire at any non-initial time $0 < t \leq T$. Short-lived contracts have payments only at the immediate successor of the issuing node, while long-lived contracts expire later and carry dividends. Let us formalize some aspects of the financial market.

Definition 33 (Financial Contract) *The financial contract $j \in \{1, ..., J\}$ issued at node $\xi(j) \in \mathbb{D}^-$ is a promise to deliver the dividend process*

$$\{V^j(\xi) \in \mathbb{R} \mid \xi \in \mathbb{D}^+(\xi(j))\}$$

for all successors of $\xi(j)$. The value of the dividends at node ξ is measured in units of good at node ξ.

Definition 34 (Actively Traded Securities) *We denote the set of actively traded financial contracts at node ξ by*

$$J(\xi) := \{j \in \{1, ..., J\} \mid \xi \in \mathbb{D}(\xi(j)),\ \exists \xi' \in \mathbb{D}^+(\xi) \text{ such that } V^j(\xi') \neq 0\}.$$

Definition 35 (Dividend Process) *The dividend process of financial contract j is* *defined as*

$$V^j := (V^j(\xi),\ \xi \in \mathbb{D}) \in \mathbb{R}^n$$

such that $V^j(\xi) = 0$ *for all* $\xi \neq \mathbb{D}^+(\xi(j))$.

Definition 36 (Price Process) *The price process of the dividends of the financial contract* j *is given by*

$$q_j = (q_j(\xi),\ \xi \in \mathbb{D}) \in \mathbb{R}^n$$

such that $q_j(\xi) = 0$ *for all* ξ*, such that* $j \notin J(\xi)$.

Definition 37 (Financial Structure) *A financial structure* (ζ, V) *for the economy* $E(\mathbb{D}, U, W)$ *with* J *financial contracts is described by the pair* (ζ, V)*, where* ζ *defines the nodes of issue of contracts*

$$\zeta = (\xi(j), j = 1, ..., J)$$

and V *is the* $n \times J$ *matrix of dividend processes*

$$V = [V^1 ... V^J].$$

Due to the fact that all the information about the financial contracts are contained in the financial structure, we obtain the financial market $E(\mathbb{D}, U, W, \zeta, V)$ by endowing the contingent market with a financial structure.

Definition 38 (Space of Portfolios) *Let* $z^i(\xi) = (z_1^i(\xi), ..., z_J^i(\xi)) \in \mathbb{R}^J$ *be the portfolio of agent* i *at node* ξ*. We define the space of portfolios as*

$$\mathbb{Z} = \{z \in \mathbb{R}^{(\#\mathbb{D}^-) \times J} \mid z = (z_1(\xi), ..., z_J(\xi), \xi \in \mathbb{D}^-, z_j(\xi) = 0 \ \textit{if}\ j \notin J(\xi)\}.$$

For a given wealth endowment vector, the choice of a portfolio for agent i brings up the following budget equations that must be satisfied by the allocation x^i:

$(C1)$ $x^i(\xi_0) - w^i(\xi_0) = -q(\xi_0)z^i(\xi_0)$

$(C2)$ $x^i(\xi) - w^i(\xi) = (V(\xi) + q(\xi))z^i(\xi^-) - q(\xi)z^i(\xi),\ \forall \xi \in \mathbb{D}^+ \cap \mathbb{D}^-$

$(C3)$ $x^i(\xi) - w^i(\xi) = V(\xi)z^i(\xi^-),\ \forall \xi \in \mathbb{D}_T$

Definition 39 (Payoff Matrix) *We define the payoff matrix $P(q, V)$ as the $(\#\mathbb{D}) \times (\#\mathbb{D}^-)J$-dimensional matrix that satisfies*

$$x^i - w^i = P(q, V)z^i,$$

as an equivalent matrix notation to the budget equations $(C1) - (C3)$.

An example of such a matrix would be

$$P(q, V) := \begin{bmatrix} -q(\xi_0) & 0 & 0 & 0 & 0 \\ V(\xi_0^+) + q(\xi_0^+) & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & V(\xi) + q(\xi) & -q(\xi) & 0 \\ 0 & 0 & 0 & V(\xi^+) + q(\xi^+) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where the rows represent, in order, $\xi_0,\ \xi_0^+,\ \xi,\ \xi^+$.

Definition 40 (Budget set) *We define the budget set for agent i as:*

$$\mathbb{B}(q, w^i, V) := \{x^i \in \mathbb{R}^n_+ \mid x^i - w^i = P(q, V)z^i,\ z^i \in \mathbb{Z}\}.$$

Definition 41 (Financial Market Equilibrium(FME)) *A financial market equilibrium for the stochastic finance economy $E(\mathbb{D}, U, W, \zeta, V)$ is a pair consisting of* *actions and prices* $((\bar{x}, \bar{z}), \bar{q}) \in \mathbb{R}_+^{nI} \times \mathbb{Z}^I \times \mathbb{R}^{nJ}$ *such that*

1. $(\bar{x}^i; \bar{z}^i) \in$ *argmax* $\{u^i(x^i) \mid (x^i; z^i) \in \mathbb{B}(\bar{q}, w^i, V)\},\ i = 1, ..., I,$
2. $\sum_{i=1}^{I} \bar{z}^i = 0.$

Definition 42 (Market Subspace) *We define the market subspace as the subspace* $\langle P(q, V)\rangle$ *of* $\mathbb{R}^n$ *spanned by the columns of the matrix* $P(q, V)$

$$\langle P(q, V)\rangle := \{\tau \mid \tau = Pz, z \in \mathbb{Z}\}.$$

$\langle P(q, V)\rangle$ is said to be absent of arbitrage opportunities if

$$\langle P(q, V)\rangle \cap \mathbb{R}_+^n = \{0\}.$$

The notion of completeness is similar to the one presented in Chapter 3, if linked to the maximum rank of the payoff matrix. Given that we do not allow for arbitrage opportunities, we must have at least one state-price vector $\pi \in \langle P(q, V)\rangle^{\perp}$, which gives us the maximal dimension of the market subspace equal to $n - 1$.

Definition 43 (Completeness) *Let the market subspace* $\langle P(q, V)\rangle$ *be arbitrage free. The financial market is complete if*

$$dim\langle P(q, V)\rangle = n - 1.$$

Otherwise, if

$$dim\langle P(q, V)\rangle < n - 1$$

then the market is incomplete.

Several other results about financial economics in a multi-period setting are completely analogous to the ones presented in two periods. One can define a multi-period normalized equilibrium in a similar way as in Definition 18 and then prove it is equivalent to the financial equilibrium (Section 23 in [3]). On the basis of this equivalence, the existence of financial market equilibrium with short-lived securities can be proven in an identical way as in Theorem 5. Under the assumption of completeness, the financial market is equivalent to the contingent market and any security can be replicated through contingent contracts (Proposition 22.6 in [3]).

The main difference arises when we consider an incomplete financial market with long-lived securities. First of all, such an equilibrium may not exist for a set of dividend streams and one must prove that such a set is of measure zero. Then one must consider the behaviour of endowments and dividend streams within the event-tree, but under different assumptions imposed on the rank of the payoff matrix. Such an analysis would require knowledge of degree theory and it is not attempted here, but one can find elements of a proof of existence of financial market equilibrium with long-lived securities in Section 25 of [3].

Outlook

The logical next step would be to analyze the equilibrium outcome through computing. Thus, concrete values can be obtained numerically and variations in parameters can be analyzed. For example, we can choose an explicit utility function, such as the power function $u(x,t) = \rho^t x^\beta/\beta$, where β is the risk aversion coefficient and $\rho \in (0,1)$ represents the (im)patience of the agent. Equilibrium prices can be calculated for a pair (ρ_0, β_0) and then for a different set of parameters (ρ_0, β_1) or (ρ_1, β_0), which will enable a discussion on the impact of the change in their values.

There are several ways to model the numerics of an incomplete market, one of which is given in [7]. The novelty of this algorithm is the fact that no forward iteration is needed, which reduces both the complexity of the written algorithm and the computing time. The final version is presented at page 1910 of the document [7]. Notation follows the one from [3], but we shall introduce it here.

We have the following set of equations:

1. 'marketability' condition

$$c_{l,t+1,\eta} + F_{l,t+1,\eta} = e_{l,t+1,\eta} + \theta_{l,t} \cdot (S_{t+1,\eta} + \delta_{t+1,\eta}),$$

2. 'kernel' condition

$$\frac{\mathbb{E}_t[u'_{l,t+1}(c_{l,t+1}) \times (S_{n,t+1} + \delta_{n,t+1})]}{u'_{l,t}(\omega_{l,t} \times e_t)} = \frac{\mathbb{E}_t[u'_{L,t+1}(c_{L,t+1}) \times (S_{n,t+1} + \delta_{n,t+1})]}{u'_{L,t}((1 - \sum_{l=1}^{L-1} \omega_{l,t}) \times e_t)},$$

3. market clearing condition
$$\sum_{l=1}^{L} \theta_{n,l,t} = 0$$
in the space given by

$$1 \leq l \leq L,\ \eta \in \xi^+,\ 1 \leq n \leq N,$$

with the mention that the case $l = L$ is excluded from the 'kernel' condition.

As for notation, let us give brief a explanation:

1. the subscript l stands for agent number l,
2. the subscripts t and $t+1$ stand for the time at which we calculate the given variable,
3. the subscript η represents an immediate successor of node ξ,
4. $c_{l,t+1}$ stands for the consumption of agent l at time $t+1$,
5. $u(x)$ represents the utility function and $u'_{l,t+1}(x)$ is the marginal utility of agent l at time $t+1$ given by consumption x,
6. θ are the claims on other investors in the market,
7. S are the price functions of the securities,
8. $F_{l,t+1,\eta}$ represents the future wealth function of agent l at time $t+1$, at node η,
9. δ are the dividends,
10. $\mathbb{E}_t$ is the conditional expectation with respect to time t,
11. e_t is the aggregate wealth endowment at time t, $e_t := \sum_{l=1}^{L} e_{l,t}$,
12. $\omega_{l,t}$ stands for the share of consumption of agent l at time t, $\omega_{l,t} := c_{l,t}/e_t$,

13. for further use, ϕ represents the Arrow-Debreu shadow prices.

A grid is set up for the values of ω in $(0,1)$. Then the system is solved for each pair (t,η) with the unknowns $c_{l,t+1,\eta}$ and $\theta_{l,t}$. The value of all variables, except consumption, is initialized at zero for time T.

$S_{n,t}$ and $F_{l,t}$ will require the following interpolation equations:

$$S_{n,t} = \frac{\mathbb{E}_t[\phi_{L,t1} \times (S_{n,t+1} + \delta_{n,t+1})]}{\phi_{L,t}},\ 1 \leq n \leq N,$$

$$F_{l,t} = \frac{\mathbb{E}_t[\phi_{L,t+1} \times (F_{l,t+1} + c_{l,t+1} - e_{l,t+1})]}{\phi_{L,t}},\ 1 \leq l \leq L.$$

An example of implementation would be the binomial-tree, where calculation time is proportional to the total number of nodes if the tree is recombining.

Conclusion

The theory of incomplete markets has been thoroughly examined in discrete time, for both economies of one good and several goods. The existence of a financial market equilibrium in continuous time still constitutes a challenging open problem in financial economics and it is in the focus of current academic research.

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